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Abstract

Many problems in production planning and supply chain optimization can be modeled as Economic Lot Size problem (ELS). This paper first demonstrates forward and backward algorithms for the economic lot size problem with backlogging, and then applies the two algorithms to analyze the sensitivities of setup cost and demand. The result for setup cost sensitivity is the same as the one described in [8] for economic lot size problem without backlogging. These analyses are especially useful for online or real-time applications which are common in current E_commerce.

1. Introduction

Many problems in production planning and supply chain optimization can be modeled as Economic Lot Size (ELS) problem. For example, if we consider the single product production scheduling problem at the manufacturer's site, and also consider the same product's inventory control problem at the retailer's site, then it is a typical ELS provided the production and inventory cost have linear relationship with the production and inventory units. ELS is a classical problem and first proposed in [10]. After Wagner and Whitin provided and solved the economic lot size model without backlogging by an $O(n^2)$ algorithm, considerable research efforts have been made to extend the basic model. Two important extensions are the consideration of backlogging, and the improvement of computational complexity. This paper re-interprets the algorithms given in [12] and [7], and applies the forward and backward algorithms for conducting sensitivity analysis on setup cost and demand.

2. Previous Works

Zangwill [11] considered the inventory backlogging in the basic economic lot size model, and proved that there is an optimal solution that the production and inventory at every period satisfy the *exact requirements*. In the same paper Zangwill also developed an $O(n^3)$ algorithm to solve the model. Zangwill [12] expressed the economic lot size problem with and without backlogging as a network that has one entry with the demand summations and multiple exits with the demand of every period. He also developed an efficient $O(n^2)$ algorithm for the model where the production cost is fixed. It is worthy to notice that the cost functions of every period in the Zangwill's works can be linear or nonlinear and the value can be computed in constant time. Blackburn and Kunreuther [3] developed a forward algorithm for the case where the production, holding and backlogging cost functions are linear. Some breakthrough was made at the beginning of 1990's. Wagelmans *et al.* [9] reformulate Wagner-Whitin model into a new model without holding cost, and employ a geometric method to reduce the computational complexity. Furthermore Hoesel, Wagelmans and Moerman [7] proved that the economic lot size problem with backlogging can also be solved with the same computational complexity as $O(n\log n)$ using geometric methods. Federgruen and Tzur [5] [6] also develop an $O(n\log n)$ forward dynamic algorithm to determine the optimal last setup period and minimum cost sequentially for the ELS with and without backlogging. Aggarwal and Park [2] identify the connections of the known fast array searching techniques in Monge arrays with the ELS, and obtain the same results.

Based on the geometric interpretation of the ELS developed in [9], Hoesel and Wagelmans [8] studied the variation scopes of setup, production, holding costs and demand respectively such that the optimal production schedule remained unchanged. If the setup cost for every period is constant and the unit production cost is also fixed, Chand and Voros [4] prove that the total cost of holding and backlogging is a non-increasing convex function of the number of setups for the ELS with and without backlogging, and develop an $O(n^2)$ forward algorithm to compute the stability region of the setup cost.

3. The Notations

Mathematically, economic lot sizing model with backlogging is to satisfy the known demands in a planning horizon at a minimum cost by a single commodity. ELS with backlogging permits to produce later to satisfy the demand of a former period. The following notations will be used through this article:

N: the length of the planning horizon,
d_i: the demand in period $i \in \{1, \dots, N\}$,
p_i: the unit production cost in period $i \in \{1, \dots, N\}$,
f_i: the setup cost in period $i \in \{1, \dots, N\}$,
h_i^+: the unit inventory cost $i \in \{1, \dots, N\}$,
h_i^-: the unit backlogging cost $i \in \{1, \dots, N\}$,

$d_{i,j} = \sum_{t=i}^{j} d_t$, $1 \le i \le j \le N$,

$h^+_{i,j} = \sum_{t=i}^{j} h^+_i$, $1 \le i \le j \le N$,

$h^-_{i,j} = \sum_{t=i}^{j} h^-_i$, $1 \le i \le j \le N$,

$F(i)$: the minimum cost from period 1 to the end of period *i,* the demands are satisfied by the productions in these periods. The inventory at the end of period is zero (such period is defined as regeneration period).

$F'(i)$: the minimum cost from period 1 to the end of period *i,* the demands are satisfied by the productions in these periods. Period *i* is a production period (Such period is defined as production period).

$B(i)$: the minimum cost from the beginning of period *i* to the end of period *N*, the demands are satisfied by the productions in these periods. The inventory at the beginning of period is zero.

$B'(i)$: the minimum cost from the beginning of period *i* to the end of period *N*, the demands are satisfied by the productions in these periods. Period *i* is a production period.

4. Forward and Backward Algorithms

Zangwill [12] gave a two-step expression to demonstrate ELS with backlogging. Hoesel, Wagelmans and Moerman [7] reformulated the formulas into the ones that can be explained in a geometric approach. Based on both the preceding works, this article re-formulates the backward and forward dynamic expressions with the parameters within the studied period scope, keeps the formulas' geometric meaning, and subsequently applies them in the setup cost sensitivity analysis and total demand variation.

4.1 Backward Algorithm

In [12] and [7], the backward algorithm of ELS with backlogging can be expressed as the following equations:

$$B(s) = \min_{s \le t \le N+1} \{p_t d_{s,t-1} + \sum_{\tau=s}^{t-1} h^-_\tau d_{s,\tau} + B'(t)\} \qquad (1)$$

$$B'(t) = \min_{t<u\le N+1} \{f_t + p_t d_{t,u-1} + \sum_{\tau=t}^{u-2} h^+_\tau d_{\tau+1,u-1} + B(u)\} \qquad (2)$$

and define $B(N+1) = 0$, $B'(N+1) = 0$.

Define $M(s, t) = \sum_{\tau=s}^{t-1} h^-_\tau d_{s,\tau}$ and

$$M'(t,u) = \sum_{\tau=t}^{u-2} h^+_\tau d_{\tau+1,u-1}$$

The following equalities hold:

$M(s, t) = M(s,N) - M(t,N) - h^-_{t,N-1} d_{s,t-1}$,

$M'(t,u) = M'(t,N) - M'(u,N) - h^+_{t,u-1} d_{u,N-1}$.

So the equalities (1) and (2) can be re-written as:

$$\begin{aligned} B(s) &= \min_{s\le t\le N}\{p_t d_{s,t-1} + M(s,N) - M(t,N) - h^-_{t,N-1} d_{s,t-1} + B'(t)\} \\ &= M(s,N) + \min_{s\le t\le N}\{B'(t) - M(t,N) - (p_t - h^-_{t,N-1}) d_{t,N} + (p_t - h^-_{t,N-1}) d_{s,N}\} \end{aligned} \qquad (3)$$

$$\begin{aligned} B'(t) &= \min_{t<u\le N+1}\{f_t + p_t d_{t,u-1} + M'(t,N) - M'(u,N) - h^+_{t,u-1} d_{u,N-1} + B(u)\} \\ &= M'(t,N) + f_t - h^+_{t,N} d_{t,N-1} + \min_{t<u\le N+1}\{B(u) - M'(u,N) + h^+_{u,N} d_{u,N-1} + (p_t + h^+_{t,N}) d_{t,u-1}\} \end{aligned} \qquad (4)$$

The geometric meaning of equalities (3) and (4) can be explained as follows: The lines passing through (0, $B'(t) - M(t,N) - (p_t - h^-_{t,N-1}) d_{t,N}$) with slopes ($p_t - h^-_{t,N-1}$) construct the concave lower envelope, the $B(s)$ can be obtained by maintaining this concave lower envelope; the points of ($d_{t,N}$, $B(u) - M'(u,N) + h^+_{u,N} d_{u,N-1}$) construct a convex lower envelope, $B'(t)$ can be obtained by searching the point that the line passing through with slope $p_t + h^+_{t,N}$ is tangent to the envelope. (Please refer to [7] for detail).

4.2 Forward Algorithm

The forward algorithm can be expressed as the following formulas:

$$F(s) = \min_{0\le t\le s}\{p_t d_{t+1,s} + \sum_{\tau=t}^{s-1} h^+_\tau d_{\tau+1,s} + F'(t)\} \qquad (5)$$

$$F'(t) = \min_{0\le u<t}\{f_t + p_t d_{u+1,t} + \sum_{\tau=u+1}^{t-1} h^-_\tau d_{u+1,\tau} + F(u)\} \qquad (6)$$

and define $F(0) = 0$, $F'(0) = 0$.

Similarly, define

$m(t, s) = \sum_{\tau=t}^{s-1} h^+_\tau d_{\tau+1,s}$, and

$m'(u,t) = \sum_{\tau=u+1}^{t-1} h^-_\tau d_{u+1,\tau}$.

The following equalities hold:

$m(t,s) = m(1,s) - m(1,t) - h^+_{1,t-1} d_{t+1,s}$,

$$m'(u,t) = m'(1,t) - m'(1,u) - h^{-}_{u,t-1} d_{2,u}$$

Equalities (5) and (6) can be rewritten as following:

$$F(s) = \min_{0 \le t \le s} \{ p_t d_{t+1,s} + m(1,s) - m(1,t) - h^{+}_{1,t-1} d_{t+1,s} + F'(t) \}$$
$$= m(1,s) + \min_{0 \le t \le s} \{ F'(t) - m(1,t) - (p_t - h^{+}_{1,t-1}) d_{1,t} + (p_t - h^{+}_{1,t-1}) d_{1,s} \} \quad (7)$$

$$F'(t) = \min_{0 \le u < t} \{ f_t + p_t d_{u+1,t} + m'(1,t) - m'(1,u) - h^{-}_{u,t-1} d_{2,u} + F(u) \}$$
$$= f_t + m'(1,t) - h^{-}_{1,t-1} d_{2,t} + \min_{0 \le u < t} \{ F(u) - m'(1,u) + h^{-}_{1,u-1} d_{2,u} + (p_t + h^{-}_{1,t-1}) d_{u+1,t} \} \quad (8)$$

The similar geometric meaning of equalities (7) and (8) can be explained as follows: The lines passing through (0, $F'(t) - m(1,t) - (p_t - h^{+}_{1,t-1}) d_{1,t}$) with slopes ($p_t - h^{+}_{1,t-1}$) construct the concave lower envelope, the $F(s)$ can be obtained by maintaining this concave lower envelope; the points of ($d_{1,u}$, $F(u) - m'(1,u) + h^{-}_{1,u-1} d_{2,u}$) construct a convex lower envelope, $F'(t)$ can be obtained by searching the point that the line passing through with slope $p_t + h^{-}_{1,t-1}$ is tangent to the envelope.

Now we have the backward and forward dynamic algorithms that are expressed with the parameters within the study periods.

5. Setup Cost Sensitivity Analysis

In this section, two scenarios will be considered: setup cost at a period is decreased, or increased by a variation of δ. The objective is to get the variation scope of the δ such that the production schedule remains unchanged.

Scenario 1: Setup cost is decreased from f_i to $f_i - \delta$.

First suppose period *i* is a production period in the original problem and we will prove the following proposition.

Proposition 1: If Period *i* is a production period in ELS model with backlogging, then period *i* will still be a production period if its setup cost is decreased.

Proof: Suppose the optimal production schedule S_1 for the new problem does not include period *i.* The minimum cost of the new optimal production schedule will not be less than that one for the original problem which period *i* is a production period. This contradicts that S_1 is the optimal production schedule.□

Lemma 1: If period *i* is a production period in the final optimal production schedule, then

$$B(1) = F(N) = F'(i) + B'(i) - f_i - p_i d_i .$$

Proof: Suppose the final production schedule is: $1 \le i_1 < i_2 < \dots i_j < i < i_r < \dots < i_k \le N$. If we can prove that the production schedule before and after period *i* in this final schedule is the optimal production schedule for $F'(i)$ and $B'(i)$ respectively, we prove the lemma. This can be proved by contradiction.

Suppose there is a better production schedule $i_{1'}, i_{2'}, \dots, i_{j'}, i$ other than $i_1, i_2, \dots, i_j, i$ for $F'(i)$ which the total cost for $F'(i)$ is smaller, then obviously $i_{1'}, i_{2'}, \dots, i_{j'}, i, i_r, \dots, i_k$ will be a better production schedule than $i_1, i_2, \dots, i_j, i, i_r, \dots, i_k$. This is a contradiction to our supposition. In similar way, we can prove $i, i_r, \dots, i_k$ is the optimal production schedule for $B'(i)$. Because $f_i + p_i d_i$ is counted twice together in $F'(i)$ and $B'(i)$, so we have $B(1) = F(N) = F'(i) + B'(i) - f_i - p_i d_i$.□

It is obvious to have the following lemma directly from equalities (4) and (8):

Lemma 2: If in period *i*, the setup cost is decreased from f_i to $f_i - \delta$ (period *i* might not be a production period), then

(1) From period 1 to *i*-1, $F'(.)$ remains unchanged; $F'(i)$ decreases by δ;

(2) From period i+1 to *N*, $B'(.)$ remains unchanged; $B'(i)$ decreases by δ.

From Proposition 1 and preceding analysis, we can get the following conclusion:

Proposition 2: If period *i* is a production period in ELS with backlogging, and its setup cost is decreased from f_i to $f_i - \delta$, and the production schedule remains unchanged, then the optimal cost decreases by δ.

Now suppose period *i* is not a production period in the original problem. Suppose period *i* becomes a production period in the new problem for its setup cost decreases from f_i to $f_i - \delta$, then the optimal cost is $F'(i) + B'(i) - f_i - p_i d_i - \delta$, and this value should be at least less than the original minimum cost. So:

$$F'(i) + B'(i) - f_i - p_i d_i - \delta < F(1),$$

then $\delta > F'(i) + B'(i) - f_i - p_i d_i - F(1)$.

So if $\delta \le F'(i) + B'(i) - f_i - p_i d_i - F(1)$, then period *i* will not become a production period in the new problem. So δ is bounded by min{ f_i , $F'(i) + B'(i) - f_i - p_i d_i - F(1)$ }. So the following conclusion can be obtained:

Proposition 3: If Period i is not a production period in ELS with backlogging, and its setup cost is decreased from f_i to $f_i - \delta$, the variation of δ is bounded by Min{ f_i, $F'(i) + B'(i) - f_i - p_i d_i - F(1)$ }.

Scenario 2: Setup cost is increased from f_i to $f_i + \delta$.

Opposite to Proposition 1, it is obvious that if a period is not a production period in the final production schedule, it is impossible for it to become a production period if its setup cost is increased. Thus, the period is a production period in the final production schedule can be considered only.

The idea here is to compute the optimal minimum value for problem that period i is not a production period from period 1 to N, the obtained optimal value is the upper bound for the problem that the setup cost of period i is increased and period i remains a production period.

Mathematically, the optimal minimum cost can be expressed as:

$$G = \min_{1 \le j < i < t \le N} \{F'(j) + g(j,t) + B'(t)\}, \tag{9}$$

where

$$g(j,t) = \min_{j \le l \le t} \{p_j d_{j+1,l} + p_t d_{l+1,t-1} + \sum_{\tau=j}^{l-1} h_\tau^+ d_{\tau+1,l} + \sum_{\tau=l+1}^{t-1} h_\tau^- d_{l+1,\tau}\}$$

In order to get the value of $G(.)$ in equality (9), first we prove the following lemma:

Lemma 3: If a period is not a production period in the final production schedule, and its setup cost is increased, this period will still not be a production period in the final production schedule, and the final production schedule and optimal cost will remain unchanged.

Proof: Similar to proposition 1, the first part is obvious. The second part is proved as follows:

Suppose setup cost at period i is $f_i + \delta_1$ and $f_i + \delta_2$ respectively, and period i is not a production period in the final production schedule for both problems with different setup cost. Because the optimal total cost will only count the setup costs of final production periods, holding and backlogging costs between the consecutive production periods, and the other parameter values except the setup cost at period i are the same for both problems, so a feasible solution for one problem is also feasible for the other one. This proves the lemma. □

Lemma 3 will be used in the following idea to compute the value of $G(.)$: Because of increasing setup cost, it is possible that a production period i will be out of the final production schedule. So there is a break value of setup cost that period i is a production period in the final production schedule if its setup cost is less than this value (Proposition 1), and period i is not a production period in the final production schedule if its setup cost is larger than this value (Lemma 3). Also from Lemma 3, if setup cost is larger than the setup break point value, the optimal final cost is the same no matter how big the setup cost is. So period i in equality (9) can be got if a big enough setup cost is assigned to period i such that period i will not be a production period in the final optimal solution.

Now it is clear that we can get the value of G in $O(n\log n)$ if the setup cost for period i is assigned a big enough value so that the period i is not a production period in the final solution. Of course we can start to solve the problem from $B(i+1)$ or $F(i-1)$. Such big enough setup value for period i exists, for example the new setup cost is:

$$f_i = \sum_{j=1}^{N} f_j + d_{1,N}[N * Max(h_1^-, h_2^-, ..., h_N^-) + N * Max(h_1^+, h_2^+, ..., h_N^+) + Max(p_1, p_2, ...p_N)$$

If production occurs at such a period, the cost will be more than that for setup at every period, carrying and backlogging the total demands from the beginning to the end, and producing at the most expensive unit cost. Any production schedule other than including this period will definitely be smaller.

After value of G is computed, the period i will remain the production period if $F(n) + \delta \le G$. So the upper bound for the value of δ is $G-F(n)$. So the following result holds:

Proposition 4: The maximal allowable increase of f_i can be calculated in $O(n\log n)$ time.

6. Total Demand Variation Analysis

It is also obvious to get the following lemma from equalities (4) and (8):

Lemma 4: If in period i, the demand is changed from d_i to $d_i \pm \delta$ then

(1) From period 1 to i-1, $F'(.)$ remains unchanged, $F'(i)$ changes by $\pm p_i\delta$;

(2) From period i+1 to N, $B'(.)$ remains unchanged, $B'(i)$ changes by $\pm p_i\delta$.

Here, we will study how the variation of the total demand affects the original optimal production plan.

Suppose the $\pm\delta$ units are increased or decreased at period i. The objective is still to let the whole costs minimum. Such minimum value can be expressed mathematically as follows for the case the products are increased or decreased:

$$VI(\delta) = \min_{1 \le i \le N}\{F'(i) + B'(i) - f_i - p_i d_i + p_i\delta\} \tag{10}$$

$$V(\delta) = \min_{1 \le i \le N}\{F'(i) + B'(i) - f_i - p_i d_i - p_i\delta\} \tag{11}$$

Let us discuss (10) first. Define l_i is the line which passes through point ($0, F'(i) + B'(i) - f_i - p_i d_i$) with slope of p_i. So lines of l_i ($1 \le i \le N$) construct a concave lower envelop for the parameter of δ. Because the values of $F'(i)+B'(i)-f_i-p_i d_i$ are the same for all production periods, so only the period with the lowest p_i can be in the concave lower envelop for all production periods. Define such period as period A. The line l_A has the minimum value of $F'(i)+B'(i)-f_i-p_i d_i$ due to A is a production period.

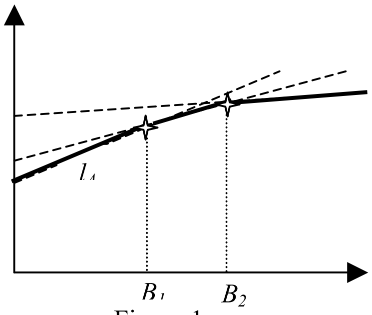

Figure 1

From Figure 1, we can see that if the increased δ is within [0, B_1], which B_1 is the first break point of the concave lower envelop, then the production period A will produce the increased units; if the increased δ is not within [0, B_1], then another period which is not a production period at the original optimal solution will produce the increased units, and therefore becomes a production period in the new solution.

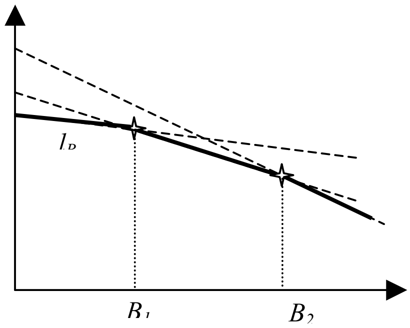

Figure 2

Same logic can applied for (11) except that l_i is the line which passes through point $(0, F'(i)+B'(i)-f_i-p_i d_i)$ with slope of $-p_i$; and only the period with the biggest p_i can be in the concave lower envelop for all production periods due to the line gradient is negative. Figure 2 describes this case.

In [7], it is proved that time complexity of constructing such a concave lower envelop in Figure 1 and 2 will be $O(n\log n)$. They also proposed that balanced tree such as 2-3 tree [1] is the efficient data structure for supporting such a procedure.

From the preceding analysis, we can draw the following result:

Proposition 5:

(1) If the solution remains the same, the increased total demand is produced at the period with least production cost; and the decreased total demand is extracted from the period with largest production cost.

(2) The maximum increased (decreased) total demand variation is the value of B_1 which is described in Figure 1 (2).

(3) The value of B_1 can be computed at time complexity of $O(n\log n)$.

7. Conclusion

This article describes the backward and forward algorithms for the economic lot size problem with backlogging, and subsequently applies the algorithms to analyze the two main parameters of the model: variation of period setup cost, and total demand variation. Although the computational complexity of the algorithm for determining the sensitivity range of setup cost and demand is $O(n\log n)$, the complexity of solving the problem itself is not augmented by sensitivity analysis since these two can be implemented concurrently. Such analysis is very useful for online or real time ELS planning or scheduling problems which are very common in today's e_commerce applications.

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